ETMAG
Lecture 6

- Functions - continued
- The limit of a function at a point

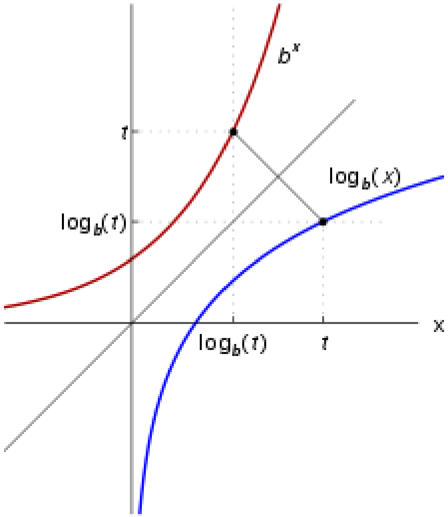
Exponential functions **should not be confused with** *power functions*.

Exponential functions have a constant base, the variable is in the exponent, as in 2^x.

In power functions the variable is in the base, the exponent is constant, as in x^2.

Fact. (Properties of powers)

- $a^b a^c = a^{b+c}$, this implies a^0=1 and $a^{-b} = \frac{1}{a^b}$
- $\left(a^b\right)^c = a^{bc}$
- $a^b c^b$=$(ac)^b$



The image from Wikipedia. In this example b>1.

Definition.

The inverse function to b^x is called the *logarithmic* function to base b and is denoted by $\log_b x$.

Fact.

- Since the domain of b^x is ℝ and the set of values is (0;∞), the domain of log_b is (0;∞) and the range is ℝ.
- b must be positive and different from 1.

Examples.

1. $log_b b = 1$ for every b for which the expression makes sense
2. $log_b 1 = 0$
3. $log_{10} 100 = 2$
4. $log_{100} 10 = 0.5$
5. $log_2 1024 = 10$

Theorem. **(**Properties of the logarithmic function)

1. if $b>1$ then $log_b x$ is increasing, if $b<1$ then $log_b x$ is decreasing
2. $b^{\log_b x} = x$ and $\log_b b^x = x$
3. $log_b xy = log_b x + log_b y$, because $b^{\log_b xy} = xy = b^{\log_b x} b^{\log_b y}$ $= b^{log_b x + log_b y}$
4. The last formula implies $log_b \frac{x}{y} = log_b x - log_b y$
5. $log_b x^a = a \, log_b x$, because $b^{log_b x^a} = x^a = \left(b^{\log_b x}\right)^a =$ $b^{a log_b x}$
6. (Change of base) $\frac{log_b x}{log_b y} = log_y x$

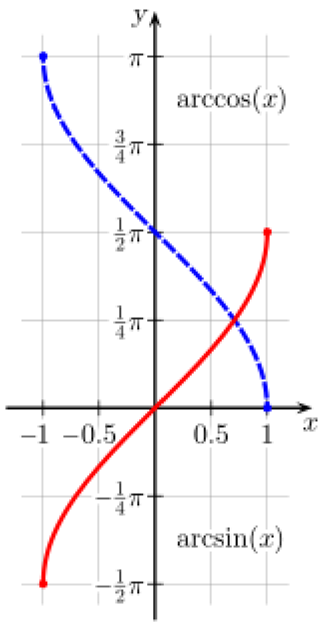
Comprehension.

1. Prove property 2
2. Prove the change of base property
3. If (a_n) is a geometric sequence what type of a sequence is $(log_b a_n)$?

Inverse trigonometric functions

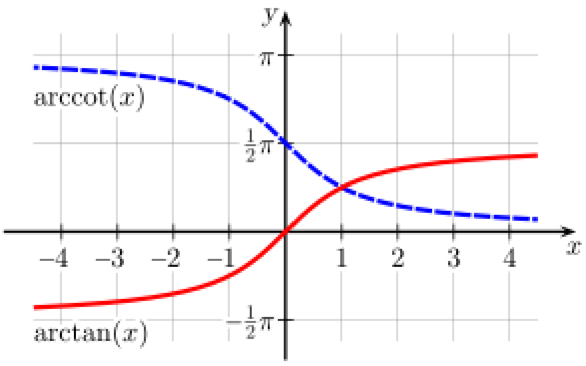
source - Wikipedia

Name	Usual notation	Definition	Domain of *x* for real result	Range of usual principal value (radians)	Range of usual principal value (degrees)
arcsine	$y = \arcsin(x)$	$x = \sin(y)$	$-1 \le x \le 1$	$-\pi/2 \le y \le \pi/2$	$-90° \le y \le 90°$
arccosine	$y = \arccos(x)$	$x = \cos(y)$	$-1 \le x \le 1$	$0 \le y \le \pi$	$0° \le y \le 180°$
arctangent	$y = \arctan(x)$	$x = \tan(y)$	all real numbers	$-\pi/2 < y < \pi/2$	$-90° < y < 90°$
arccotangent	$y = \operatorname{arccot}(x)$	$x = \cot(y)$	all real numbers	$0 < y < \pi$	$0° < y < 180°$



Graphs of arcsin and arccos

(from Wikipedia)



Graphs of arctan and arccot

(from Wikipedia)

Obviously, $\sin(\arcsin(x)) = x$, $\cos(\arccos(x)) = x$, $\tan(\arctan(x)) = x$ and $\cot(\text{arccot}(x)) = x$.

Theorem (Some relationships between trigonometric functions and cyclometric functions)

1. $\sin(\arccos(x)) = \cos(\arcsin(x)) = \sqrt{1-x^2}$

2. $\sin(\arctan(x)) = \cos(\text{arccot}(x)) = \frac{x}{\sqrt{1+x^2}}$

3. $\sin(\text{arccot}(x)) = \cos(\arctan(x)) = \frac{1}{\sqrt{1+x^2}}$

Proof.

1. Denote $\arccos(x) = t$, i.e. $\cos t = x$ and $t \in [0;\pi]$. Since $\sin t > 0$ on $[0;\pi]$ we car write $\sin t = \sqrt{1 - cos^2 t} = \sqrt{1-x^2}$.

2. Denote $\arctan(x) = t$, i.e. $\tan t = \frac{\sin t}{\cos t} = x$, which yields $\sin t = x \cos t$ for each $t \in (-\frac{\pi}{2}; \frac{\pi}{2})$. We need to express $\cos t$ in terms of x, i.e. $\tan t$. $\tan t = \frac{\sin t}{\cos t}$ implies $\mathrm{x}^2 = \tan^2 t = \frac{\sin^2 t}{\cos^2 t}$ $= \frac{1-\cos^2 t}{\cos^2 t}$ hence,

$$x^2 \cos^2 t = 1 - \cos^2 t$$

$$x^2 \cos^2 t + \cos^2 t = 1$$

$$\cos^2 t(1 + \mathrm{x}^2) = 1$$

$\cos^2 t = \frac{1}{1+\mathrm{x}^2}$. Since $t \in (-\frac{\pi}{2}; \frac{\pi}{2})$ $\cos t > 0$ and we can write $\cos t = \sqrt{\frac{1}{1+\mathrm{x}^2}} = \frac{1}{\sqrt{1+x^2}}$. Finally $\sin t = \sin(\arctan x) = \frac{x}{\sqrt{1+x^2}}$.

We can prove part 3 and other similar identities in the same way. QED

Hyperbolic functions:

- Hyperbolic sine: $\sinh x = \frac{e^x - e^{-x}}{2}$
- Hyperbolic cosine: $\cosh x = \frac{e^x + e^{-x}}{2}$
- Hyperbolic tangent: $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
- Hyperbolic cotangent: for $x \neq 0$, $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

Fact.

Just like x=cos *t* and y=sin *t* satisfy the equation of the unit circle, $x^2+y^2=1$ so x=cosh *t* and y=sinh *t* satisfy the equation of the equilateral hyperbola, $x^2-y^2 = 1$.

In fact:

$$\left(\frac{e^t+e^{-t}}{2}\right)^2 - \left(\frac{e^t-e^{-t}}{2}\right)^2 = \frac{e^{2t}+2+e^{-2t}}{4} - \frac{e^{2t}-2+e^{-2t}}{4} = \frac{4}{4} = 1$$

The limit of a function

So far, we have introduced and investigated the idea of the limit in case of sequences. Sequences are functions with the argument $n \in \mathbb{N}$, which means we study limits of rather specific functions and only when the argument approaches infinity. We will now generalize the idea to all real functions, and we will allow the variable to approach any value, infinite or not.

Let us consider a convergent sequence (a_n) with $\lim_{n\to\infty} a_n = L$ and see what happens if we apply some function *f* to all term of the sequence (assuming, of course, that all terms of our sequence belong to the domain of *f*)?

Obviously, we get a new sequence, $\left(f(a_n)\right)_{n=1}^{\infty}$ which , depending on the function *f*, may be convergent to some number or divergent to + or – infinity, or just divergent. So, the question of the limit of the sequence $\left(f(a_n)\right)$ is more about the function *f* than about the sequence (a_n).

Definition. (by Eduard Heine)
A number L is the *limit of a function at x=c* (or *as x approaches c*) iff for every sequence (x_n) convergent to c and such that for every n $x_n \neq c$, the sequence $f(x_n)$ converges to L.
In a more formal way:
$$\lim_{x\to c} f(x) = L \Leftrightarrow \left(\forall (x_n) \in (\mathbb{R} \setminus \{c\})^{\mathbb{N}}\right)\left(\lim_{n\to\infty} x_n = c \Rightarrow \lim_{n\to\infty} f(x_n) = L\right)$$
Look at the expression $(\forall (x_n) \in (\mathbb{R} \setminus \{c\})^{\mathbb{N}})$ *.
The $\forall (x_n)$ part means “for every sequence (x_n)”, (x_n) is the symbol we use for a sequence as a whole.
The $(\mathbb{R} \setminus \{c\})^{\mathbb{N}}$ denotes the set of all sequences whose terms are all different from c.

* To be honest, it should be $Dom_f \setminus \{c\}$ rather than $\mathbb{R} \setminus \{c\}$

Definition. (by Augustin-Louise Cauchy)
Let f be a real function and let $c, L \in \mathbb{R}$. We say that a number L is the *limit of* f *as x approaches* c, iff
$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in \mathbb{R})(\ 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon\ )$.
We denote this by $\lim_{x \to c} f(x) = L$.

Notice that the part "$0 < | x - c |$" means that we do not require that $c \in Dom(f)$. In fact we are NOT interested in the value of f at $x = c$, we do not even assume it exists.
This definition of the limit is called the epsilon – delta (or *Cauchy*) definition as opposed to the sequence (or *Heine*) definition.

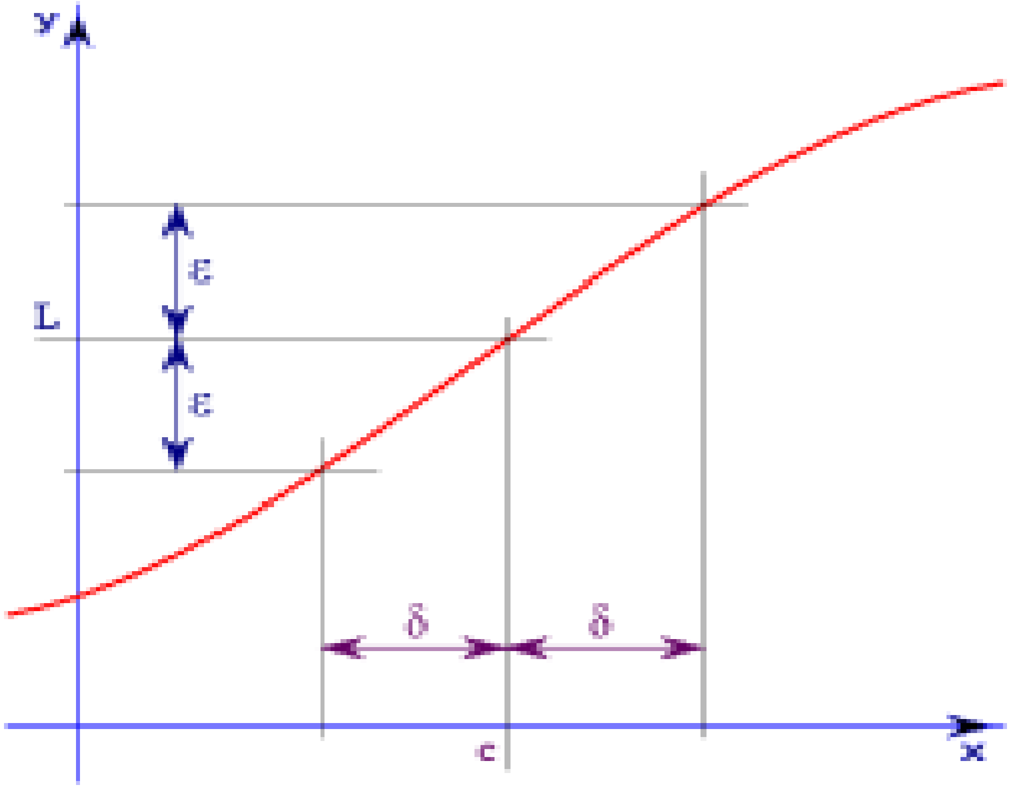


Illustration of the Cauchy definition, from Wikipedia.

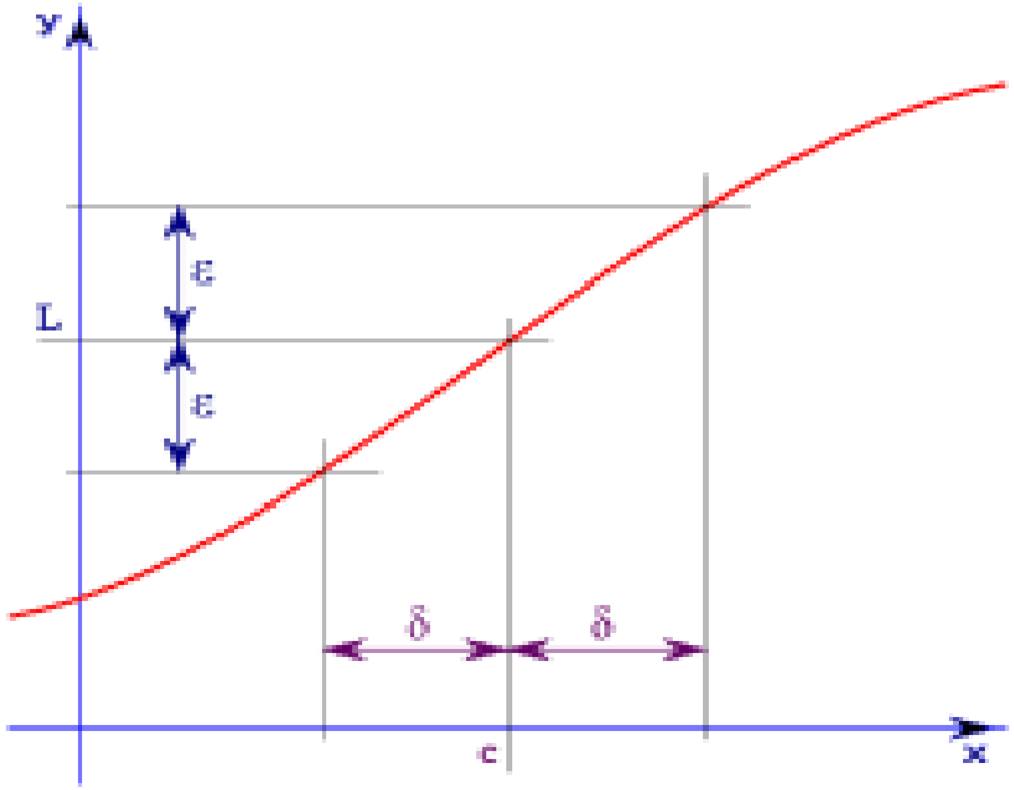
FAQ. Can we illustrate graphically the Heine definition?
Answer.
No.

But let's try anyway.

Where are n –s?

Where are x_n–s?

Where are $f(x_n)$ –s?



Theorem.
The two definitions of the limit of a function are equivalent i.e., a number L is the limit of f at some point c in the sense of Cauchy iff the number is the limit of f at c in the sense of Heine.